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Abstract

In this article we focus on instantaneous traffic flow optimization on a roundabout using a macroscopic approach. The roundabout is modeled as a concatenation of 2×2 junctions with one main lane and secondary incoming and outgoing roads. We consider a cost functional that measures the total travel time spent by drivers on the roundabout and compute its gradient with respect to the priority parameters at junctions. Then, through numerical simulations, the traffic behavior is studied on the whole roundabout. The numerical approximations compare the performance of a roundabout for instantaneous optimization of the priority parameters and fixed constant parameters.

Keywords: Traffic flow, roundabout, scalar conservation laws, gradient-based optimization.

MOS subject classification: 90B20; 35L65; 49J20

1 Introduction

The origins of macroscopic traffic flow models date back to 1950's, when the seminal works of Lighthill and Whitham [19] and Richards [21] proposed a fluid dynamic model for vehicular traffic flow on an infinite single road, commonly denoted as LWR. This model has been extended to road networks by several authors in most recent years, see [5, 6, 8, 11] and references therein. In the network setting, one-dimensional hyperbolic systems of conservation laws are also an efficient framework for modeling gas pipeline flow [1], data networks [7], and supply chains [13]. These

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models have also been utilized for the optimization of vehicular traffic flow on road networks through various approaches, see for example [3, 4, 16, 20]. Optimal control of traffic flow on networks have been investigated by different authors using different optimization techniques; detailed presentation and analysis can be found for example in [14] and reference therein.

Hyperbolic conservation laws may be nonlinear and lead to non-convex or non-linear formulations of the corresponding optimization problem. In such cases fewer optimization techniques exist for the discretized version of the problems than for convex problems. One approach is to approximate the system with a relaxed version in order to use efficient linear programming techniques. The linearization approach was used in [12] for optimal ramp metering using Godunov discretization scheme.

However, nonlinear optimization techniques such as gradient descent method can be applied to the descretized system without any modification to the underlying dynamics. Gradient descent is a first-order optimization algorithm which gives no guarantee for finding a unique global minimum in general.

The idea of instantaneous control has been discussed in [15] for the optimization of traffic flow problems on road networks. The purpose is to minimize large storage requirements arising from the network structure and the strong coupling of adjoints and state equations by generating a sequence of optimal control problems of reduced dimension. In this article, rather than generating a sequence of sub-optimal control problems, we compute the gradient of the cost function at each time step t^n to obtain the optimal priority parameters. We apply the procedure to minimize the total travel time across a roundabout. Modern roundabouts are now considered as an alternative traffic control device that can improve safety and operational efficiency at intersections when compared to other conventional intersection controls, usually for traffic flow management or to improve safety [9]. Roundabouts can be seen as particular road networks and can be modeled as a periodic concatenation of junctions. We consider a roundabout with arbitrary m incoming and m outgoing roads, $m \in \mathbb{N}$, which are described as a concatenation of m 2×2 junctions with two incoming and two outgoing roads. In particular, each junction has one incoming main lane, one outgoing main lane and a third link with incoming and outgoing fluxes. The third road is modeled by a buffer of infinite capacity for the entering flux and with an infinite sink for the exiting one. The main lane evolution is described by a scalar hyperbolic conservation law, whereas the buffer dynamics is described by an ordinary differential equation (ODE) which depends on the difference between the incoming and outgoing fluxes on the link.

The article is structured as follows. In Section 2 we describe the mathematical model for the roundabout and the governing equations. In Section 3 we present the solution of the Riemann problem at junctions. Section 4 details numerical approximations and states the update procedure. In Section 5 we describe the optimization problem of ODE-PDE constrained system and the formulation of the instantaneous flow optimization. Section 6 is devoted to numerical simulations. Finally we give conclusions in Section 7.

2 Mathematical Model for the Roundabout

In this work we consider a roundabout joining m roads, $m \in \mathbb{N}$, $m \geq 2$, as illustrated in Figure 1.

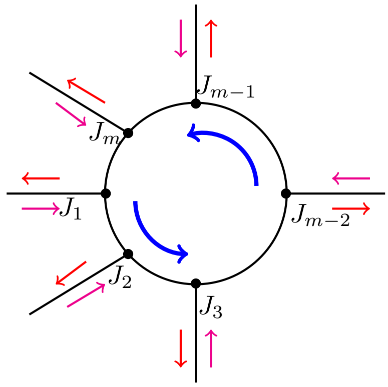

Figure 1: Sketch of the roundabout considered in the article.

A roundabout can be seen as an oriented graph in which roads are represented by arcs and junctions by vertexes. Each link forming the roundabout is modeled by intervals $I_i = [x_{i-1}, x_i] \subset \mathbb{R}$, $x_{i-1} < x_i$, $i = 1, 2, \ldots, m$. Junction J_i is located at $x = x_i$ for $i = 1, 2, \ldots, m$.

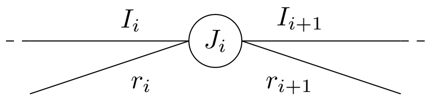

Figure 2: Sketch of a roundabout junction.

To recover the behavior of the roundabout periodic boundary conditions are introduced on the main lane such that $x_m = x_0$. Each junction is composed of a main lane and by two links representing a lane with incoming flux ($\mathrm{r_i}$) and a lane with outgoing flux ($\mathrm{r_{i+1}}$), see Figure 2. The evolution of the traffic flow on the main lane segments is given by the scalar hyperbolic conservation law:

$$\partial_t \rho_i + \partial_x f(\rho_i) = 0, \quad (t, x) \in \mathbb{R}^+ \times I_i, \quad i = 1, 2 \ldots, m, \tag{2.1}$$

where $\rho_i = \rho_i(t, x) \in [0, \rho_{\max}]$ is the mean traffic density, $\rho_{\max}$ the maximal density allowed on the road. The flux function $f : [0, \rho_{\max}] \to \mathbb{R}^+$ is given by following flux-density relation:

$$f(\rho) = \begin{cases} \rho v_{\mathrm{f}} & \text{if } 0 \leq \rho \leq \rho_{\mathrm{c}}, \\ \dfrac{f^{\max}}{\rho_{\max} - \rho_c}(\rho_{\max} - \rho) & \text{if } \rho_{\mathrm{c}} \leq \rho \leq \rho_{\max}, \end{cases}$$

with v_{f} the maximal traffic speed, $\rho_{\mathrm{c}} = \dfrac{f^{\max}}{v_{\mathrm{f}}}$ the critical density and $f^{\max} = f(\rho_{\mathrm{c}})$ the maximal flux value. We define the congested flow speed as: $w_f = \dfrac{f^{\max}}{\rho_{\max} - \rho_c}$ for $\rho_c \leq \rho \leq \rho_{\max}$. Throughout the paper, for simplicity, we will assume $\rho_{\max} = 1$ and $v_{\mathrm{f}} = 1$. Figure 3 shows an example of flux function satisfying the hypotheses. For the theory of scalar hyperbolic conservation laws we refer to [2].

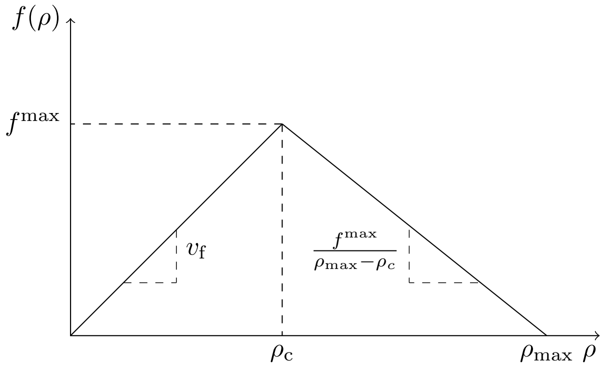

Figure 3: Flux function considered.

The incoming lanes of the secondary roads entering the junctions are modeled by buffers of infinite size and capacity. The evolution of the queue length of each buffer is described by the following ODE:

$$\frac{d}{dt}\, l_i = F_i^{\mathrm{in}} - \gamma_{\mathrm{r}_i}, \quad t \in \mathbb{R}^+, \quad i = 1, 2 \ldots m, \tag{2.2}$$

where $l_i(t) \in [0, +\infty[$ is the queue length, F_i^{in} the flux entering the lane and γ_{r_i} the flux exiting the lane. The outgoing lane is considered as a sink that accepts all the flux coming from the roundabout. No flux from the incoming lane is allowed to turn back on the outgoing stretch of the same road.
The Cauchy problem to solve is then:

$$\begin{cases} \partial_t \rho_i + \partial_x f(\rho_i) = 0, & (t,x) \in \mathbb{R}^+ \times I_i, \\ \dfrac{d}{dt}\, l_i(t) = F_i^{\mathrm{in}} - \gamma_{\mathrm{r}_i}, & t \in \mathbb{R}^+, \\ \rho_i(0,x) = \rho_i^0 & x \in I_i, \\ l_i(0) = l_{i,0} \end{cases} \tag{2.3}$$

for $i = 1, 2 \ldots, m$, where $\rho_{i,0}(x)$ are the initial densities on I_i and $l_{i,0}$ the initial lengths of the buffers. This will be coupled with an optimization problem at the junctions that gives the distribution of traffic among the roads.
We define the demand $d(F_i^{\mathrm{in}}, l_i)$ of the incoming lane for the secondary road, the demand function $\delta(\rho_i)$ on the incoming roundabout segment at each junction, and the supply function $\sigma(\rho_i)$ on the outgoing main lane segment at each junction as follows:

$$d(F_i^{\mathrm{in}}, l_i) = \begin{cases} \gamma_{\mathrm{r}_i}^{\max} & \text{if } l_i(t) > 0, \\ \min\left(F_i^{in}, \gamma_{\mathrm{r}_i}^{\max}\right) & \text{if } l_i(t) = 0, \end{cases} \tag{2.4}$$

$$\delta(\rho_i) = \begin{cases} f(\rho_i) & \text{if } 0 \le \rho_i < \rho_{\mathrm{c}}, \\ f^{\max} & \text{if } \rho_{\mathrm{c}} \le \rho_i \le 1, \end{cases} \tag{2.5}$$

$$\sigma(\rho_i) = \begin{cases} f^{\max} & \text{if } 0 \le \rho_i \le \rho_{\mathrm{c}}, \\ f(\rho_i) & \text{if } \rho_{\mathrm{c}} < \rho_i \le 1, \end{cases} \tag{2.6}$$

for $i = 1, 2, \ldots, m$, where $\gamma_{\mathrm{r}_i}^{\max}$ is the maximal flow on the incoming lane r_i, $i = 1, 2, \ldots, m$. Moreover, we introduce $\beta_i \in]0, 1[$ the split ratio of the outgoing lane r_{i+1}, and its flux $\gamma_{\mathrm{r}_{i+1}}(t) = \beta f(\rho_i(t, 0-)), i = 1, 2 \ldots, m$.

Definition 1 *Consider a roundabout as in Figure 1. A $2m$-tuple*

$$(\rho_i, l_i)_{i=1,\ldots,m} \in \prod_{i=1}^{m} \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L^1} \cap BV(\mathbb{R})\right) \times \prod_{i=1}^{m} \mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}^+; \mathbb{R}^+)$$

is an admissible solution to (2.3) if

1. *ρ_i satisfies the Kružhkov entropy condition [17] on $(\mathbb{R}^+ \times I_i)$, that is, for every $k \in \mathbb{R}$ and for all $\varphi \in \mathcal{C}_c^1(\mathbb{R} \times I_i)$, $t > 0$,*

$$\begin{aligned} &\textstyle\int_{\mathbb{R}^+} \int_{I_i} (|\rho_i - k|\partial_t \varphi + \operatorname{sgn}(\rho_i - k)(f(\rho_i) - f(k))\partial_x \varphi) dx dt \\ &\quad + \textstyle\int_{I_i} |\rho_{i,0} - k|\varphi(0, x) dx \geq 0; \quad i = 1, 2, \ldots, m. \end{aligned} \tag{2.7}$$

2. *$f(\rho_i(t, x_i-)) + \gamma_{\mathrm{r}_i}(t) = f(\rho_{i+1}(t, x_i+)) + \gamma_{\mathrm{r}_{i+1}}(t)$, $i = 1, 2, \ldots, m$.*

3. *The outgoing flux $f(\rho_{i+1}(t, x_i+))$ is maximum subject to*

$$f(\rho_{i+1}(t, x_i+)) = \min\Big((1 - \beta_i)\delta(\rho_i(t, x_i-)) + d(F_i^{\mathrm{in}}(t), l_i(t)), \sigma(\rho_{i+1}(t, x_i+))\Big), \tag{2.8}$$

   and 2.

4. *l_i solves* (2.2) *for almost every $t \in \mathbb{R}^+$.*

Remark 1 *A parameter $q_i \in]0, 1[$ is introduced to ensure uniqueness of the solution, that is, q_i is a priority parameter that defines the amount of flux that enters the outgoing mainline from each incoming road. In particular, when the priority applies, $q_i f(\rho_{i+1}(t, x_{i+1/2}-))$ is the flux allowed from the incoming mainline into the outgoing mainline, and $(1 - q_i) f(\rho_{i+1}(t, x_{i+1/2}-))$ the flux from the onramp.*

3 Riemann Problem at Junction

In this section we recall briefly the construction of the Riemann solver at a junction following [8], with modification on Γ_i in order to meet priority conditions on the roundabout. We define the Riemann Solver at junction by means of a Riemann Solver $\mathcal{RS}_{\bar{l}} : [0, 1]^2 \to [0, 1]^2$, which depends on the instantaneous load of the buffer $\bar{l}$. For each $\bar{l}$, the Riemann Solver $\mathcal{RS}_{\bar{l}}(\rho_i, \rho_{i+1}) = (\hat{\rho}_i, \hat{\rho}_{i+1})$ is constructed in the following way. For each $i = 1, \ldots, m$, we fix $q_i \in [0, 1]$ and proceed:

1. Define $\Gamma_i = f(\rho_i(t, x_i-))$, $\Gamma_{i+1} = f(\rho_{i+1}(t, x_i+))$, $\Gamma_{\mathrm{r}_i} = \gamma_{\mathrm{r}_i}(t)$;
2. Consider the space $(\Gamma_i, \Gamma_{\mathrm{r}_i})$ and the sets $\mathcal{O}_i = [0, \delta(\rho_i)]$, $\mathcal{O}_{\mathrm{r}_i} = [0, d(F_i^{\mathrm{in}}, \bar{l})]$;
3. Trace the lines $(1 - \beta_i)\Gamma_i + \Gamma_{\mathrm{r}_i} = \Gamma_{i+1}$; and $\Gamma_i = \dfrac{q_i}{(1 - q_i)(1 - \beta_i)} \Gamma_{\mathrm{r}_i}$;
4. Consider the region

$$\Omega_i = \Big\{(\Gamma_i, \Gamma_{\mathrm{r}_i}) \in \mathcal{O}_i \times \mathcal{O}_{\mathrm{r}_i} : (1 - \beta_i)\Gamma_i + \Gamma_{\mathrm{r}_i} \in [0, \Gamma_{i+1}]\Big\}. \tag{3.1}$$

Different situations can occur depending on the value of Γ_{i+1}:

- **Demand-limited case:** $\Gamma_{i+1}=(1-\beta_i)\delta(\rho_i)+d(F_i^{\mathrm{in}},\bar{l})$.
  We set $\hat{\Gamma}_i=\delta(\rho_i)$, $\hat{\Gamma}_{\mathrm{r}_i}=d(F_i^{\mathrm{in}},\bar{l})$ and $\hat{\Gamma}_{i+1}=(1-\beta_i)\delta(\rho_i)+d(F_i^{\mathrm{in}},\bar{l})$, as illustrated in Figure 4(a).
- **Supply-limited case:** $\Gamma_{i+1}=\sigma(\rho_{i+1})$.
  We set Q to be the intersection point of $(1-\beta_i)\Gamma_i+\Gamma_{\mathrm{r}_i}=\Gamma_{i+1}$ and $\Gamma_i=\dfrac{q_i}{(1-q_i)(1-\beta_i)}\Gamma_{\mathrm{r}_i}$.
  If $Q\in\Omega_i$, we set $(\hat{\Gamma}_i,\hat{\Gamma}_{\mathrm{r}_i})=Q$ and $\hat{\Gamma}_{i+1}=\Gamma_{i+1}$, see Figure 4(b); if $Q\notin\Omega_i$, we set $(\hat{\Gamma}_i,\hat{\Gamma}_{\mathrm{r}_i})=S$ and $\hat{\Gamma}_{i+1}=\Gamma_{i+1}$, where S is the point of the segment $\Omega_i\cap(\Gamma_i,\Gamma_{\mathrm{r}_i}):(1-\beta_i)\Gamma_i+\Gamma_{\mathrm{r}_i}=\Gamma_{i+1}$ closest to the line $\Gamma_i=\dfrac{q_i}{(1-q_i)(1-\beta_i)}\Gamma_{\mathrm{r}_i}$ see Figure 4(c).

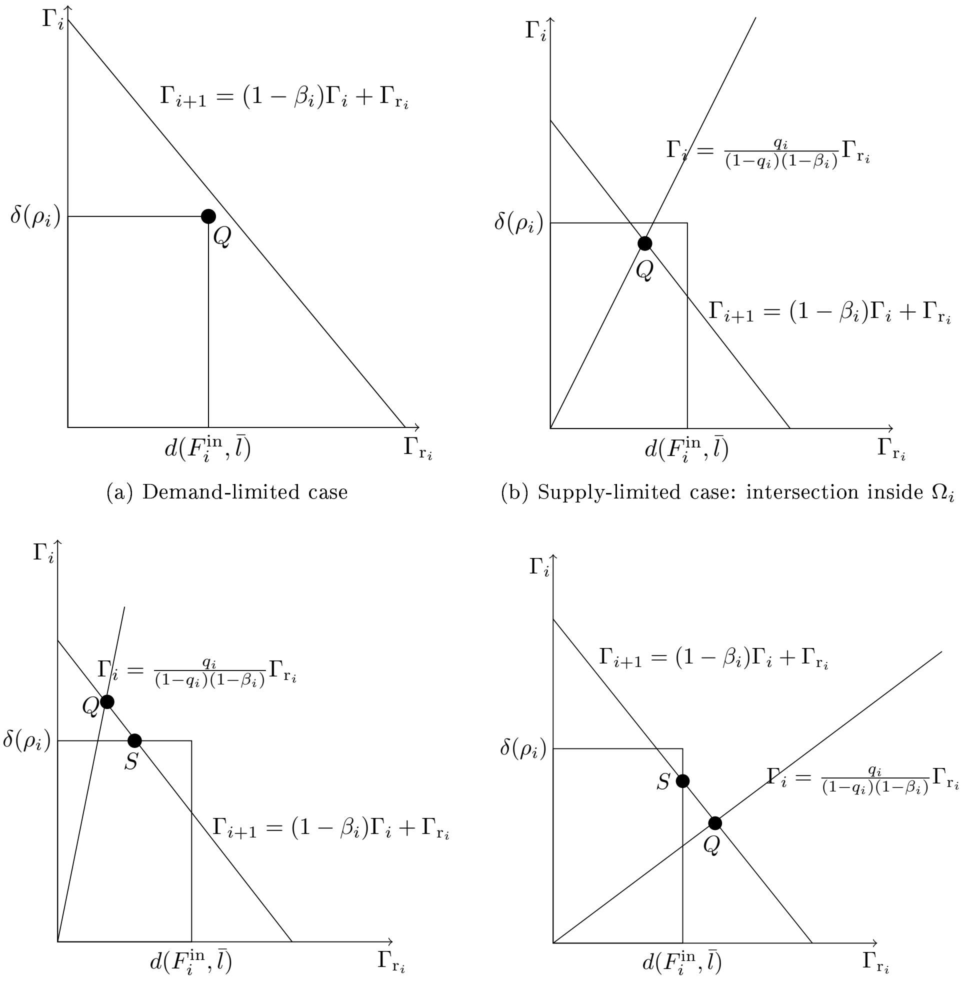

Figure 4: Solutions of the Riemann Solver at the junction.

We define the function τ as follows, for details see [10].

Definition 2 *Let* $\tau:[0,1]\to[0,1]$ *be the map such that:*

- $f(\tau(\rho)) = f(\rho)$ *for every* $\rho \in [0,1]$*;*
- $\tau(\rho) \neq \rho$ *for every* $\rho \in [0,1] \setminus \{\rho_c\}$.

Theorem 1 *Consider a junction J_i and fix a priority parameter $q_i \in]0,1[$. For every $\rho_i^0, \rho_{i+1}^0 \in [0,1]$ and $l_i^0 \in [0,+\infty[$ there exists a unique amdmissible solution $(\rho_i(t,x), \rho_{i+1}(t,x), l_i(t))$ satisfying the priority (possibly in an approximate way). Moreover, there exists a unique couple $(\hat{\rho}_i, \hat{\rho}_{i+1}) \in [0,1]^2$ such that*

$$\hat{\rho}_i \in \begin{cases} \{\rho_i^0\} \cup]\tau(\rho_i^0), 1] & \text{if } 0 \leq \rho_i^0 \leq \rho_c, \\ [\rho_c, 1] & \text{if } \rho_c \leq \rho_i^0 \leq 1, \end{cases} \qquad f(\hat{\rho}_i) = \hat{\Gamma}_i, \tag{3.2}$$

and

$$\hat{\rho}_{i+1} \in \begin{cases} [0, \rho_c] & \text{if } 0 \leq \rho_{i+1}^0 \leq \rho_c, \\ \{\rho_{i+1}^0\} \cup [0, \tau(\rho_{i+1}^0)[& \text{if } \rho_c \leq \rho_{i+1}^0 \leq 1, \end{cases} \qquad f(\hat{\rho}_{i+1}) = \hat{\Gamma}_{i+1}. \tag{3.3}$$

For the incoming road the solution is given by the wave $(\rho_i^0, \hat{\rho}_i)$, while for the outgoing road the solution is given by the wave $(\hat{\rho}_{i+1}, \rho_{i+1}^0)$. Furthermore, for almost every $t > 0$, it holds

$$(\rho_i(t, x_i-), \rho_{i+1}(t, x_i+)) = \mathcal{RS}_{l_i(t)}(\rho_i(t, x_i-), \rho_{i+1}(t, x_i+)).$$

For the proof we refer the reader to [8].

4 Numerical Approximation

To compute approximate solutions, we adapt the classical Godunov scheme to our problem with some adjustment due to the presence of the buffer as detailed in Section 4.3. We use the following notation to define a numerical gride in $(0,T) \times [x_0, x_m]$:

- x_i, $i = 0, \ldots, m$, are the cell interfaces;
- $L_i = x_i - x_{i-1}$ represents the length of each segment of the roundabout and Δt is the time step, to be defined according the CLF stability condition below;
- $t^n = t^{n-1} + \Delta t$, $n \in \mathbb{N}$ are the time grid points.

4.1 Godunov Scheme at Junctions

In this work we consider the distance between two adjacent junctions of a roundabout as the fixed space grid size of the Godunov discretization and junctions as cell interfaces. This is because each link of the roundabout is reasonably short to fit the scheme discretization. In this setting, on the main road of the roundabout we set

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{L_i}\left(F_i^{n,-} - F_{i-1}^{n,+}\right), \quad i = 1, 2, \ldots, m, \tag{4.1}$$

where

$$\begin{aligned} F_i^{n,-} &= F_i^{n,-}\left(\rho_i^n, \rho_{i+1}^n, q_i, l_i, F_i^{\text{in}}\right) \\ F_{i-1}^{n,+} &= F_{i-1}^{n,+}\left(\rho_{i-1}^n, \rho_i^n, q_{i-1}, l_{i-1}, F_{i-1}^{\text{in}}\right), \end{aligned} \qquad i = 1, 2, \ldots, m. \tag{4.2}$$

The fluxes $F_{i-1}^{n,+}$ and $F_i^{n,-}$ stand respectively for inflow $\hat{\Gamma}_i$ and outflow $\hat{\Gamma}_{i+1}$ at junctions J_{i-1} and J_i on the main road of the roundabout.

Under the Courant-Friedrichs-Lewy (CFL) condition [18]

$$\Delta t \leq \frac{\min_i L_i}{\lambda^{\max}} \tag{4.3}$$

where $\lambda^{\max} = \max\{v_f, w_f\}$. This condition ensures that waves origination at an interface do not cross other interfaces before Δt.

4.2 Computing Buffer Length From ODE

We consider the buffer evolution described by (2.2) to compute the queue length on the secondary road of a roundabout. At each time step $t^n = t^{n-1} + \Delta t$ we update the new value of the queue length as follows:

1. If $F_i^{\text{in}} \geq \gamma_{\text{r}_i}$, the buffer length is increasing and we set

$$l_i^{n+1} = l_i^n + \left(F_i^{\text{in}} - \gamma_{\text{r}_i}\right)\Delta t, \quad i = 1, 2, \ldots, m. \tag{4.4}$$

2. If $F_i^{\text{in}} < \gamma_{\text{r}_i}$, the buffer length is decreasing and we set

$$\tilde{t}_i = \frac{l_i^n}{\gamma_{\text{r}_i} - F_i^{\text{in}}} + t^n \tag{4.5}$$

the time at which the buffer empties.

i. if $t^{n+1} < \tilde{t}_i$

$$l_i^{n+1} = l_i^n + \left(F_i^{\text{in}} - \gamma_{\text{r}i}\right)\Delta t, \quad i = 1, 2, \ldots, m; \tag{4.6}$$

ii. if $\tilde{t}_i < t^{n+1}$, $\widetilde{\Delta t} = \tilde{t} - t^n$ and $l_i^{n+1} = 0$.

4.3 Adapted Godunov Scheme

We detail here the scheme used for numerical computations. We update the values of optimal priority parameter depending on the situation at junction of a roundabout as detailed in Section 5 and the state variables as follows:

Algorithm 1: State variable update procedure

1. Input: states at time t^n: $(\rho_i^n, l_i^n)_{i=1,\ldots,m}$.
2. Output: states at time t^{n+1}: $(\rho_i^{n+1}, l_i^{n+1})_{i=1,\ldots,m}$.
   (a) If $\tilde{t}_i > t^{n+1}$ or $\tilde{t}_i \leq t^n$ $\forall i$: set $t^{n+1} = t^n + \Delta t$ and update the density on roundabout with Godunov fluxes at junctions

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{L_i}\left(F_i^{n,-} - F_{i-1}^{n,+}\right), \quad i = 1, 2, \ldots, m,$$

   and the buffer lengths at junction J_i of the roundabout with:

$$l_i^{n+1} = l_i^n + \Delta t\left(F_i^{\text{in}} - \gamma_{\text{r}_i}\right), \quad i = 1, 2, \ldots, m.$$

(b) If $t^n < \tilde{t}_i < t^{n+1}$, $\exists i \in \{1,2,\dots,m\}$: set $t^{n+1} = \tilde{t}_i$, and $\widetilde{\Delta t} = t^{n+1} - t^n$; update the density on the roundabout with Godunov fluxes at junctions

$$\rho_i^{n+1} = \rho_i^n - \frac{\widetilde{\Delta t}}{L_i}\left(F_i^{n,-} - F_{i-1}^{n,+}\right), \quad i = 1,2,\dots,m,$$

and the buffer lengths

$$l_i^{n+1} = l_i^n + \left(F_i^{\text{in}} - \gamma_{\text{r}_i}\right)\widetilde{\Delta t}, \quad i = 1,2,\dots,m.$$

This Algorithm takes as inputs the states ρ_i^n, and l_i^n at time-step n for all roundabout links and on the incoming secondary roads and returns the states advanced by one time step.

Below, we explicit $F_i^{n,\pm}$ and γ_{r_i} based on different cases of Riemann Solvers at junction. The situation at junctions can be demand-limited or supply-limited. Both cases are detailed.

(a) **Demand-limited case at junction** J_i**:** $(1-\beta_i)\delta(\rho_i^n) + d(F_i^{\text{in}}, l_i^n) \le \sigma(\rho_{i+1}^n)$

In this case we set $\hat{\Gamma}_i = \delta(\rho_{i-1})$ and $\gamma_{\text{r}_i} = d(F_i^{\text{in}}, l_i)$. From this it is evident that those who demand to enter the junction can assess it without restriction. Hence,

$$F_i^{n,+} = (1-\beta)\delta(\rho_i^n) + d(F_i^{\text{in}}, l_i^n) \tag{4.7}$$

and at the instant time t^n, the flux at the left side of the junction reads as

$$F_i^{n,-} = \delta(\rho_i^n). \tag{4.8}$$

Besides, the dynamics of buffer length at the entrance of the roundabout is governed by:

$$\frac{d}{dt}\, l_i^n(t) = F_i^{\text{in}} - d(F_i^{\text{in}}, l_i^n),$$

which gives

$$l_i^{n+1} = l_i^n + \left(F_i^{\text{in}} - d(F_i^{\text{in}}, l_i^n)\right)\Delta t. \tag{4.9}$$

(b) **Supply-limited case at Junction** J_i**:** $(1-\beta_i)\delta(\rho_i^n) + d(F_i^{\text{in}}, l_i^n) > \sigma(\rho_{i+1}^n)$

We calculate the flux at the right hand side of the junction as

$$F_i^{n,+} = \sigma(\rho_{i+1}^n), \tag{4.10}$$

while on the left side it becomes

$$F_{i+\frac{1}{2}}^{n,-} = \begin{cases} \delta(\rho_i^n) & \text{if } 1 \ge q_i > \min\{1, Q_2^i\}, \\ \dfrac{q_i}{1-\beta_i}\sigma(\rho_{i+1}^n) & \text{if } \max\{0, Q_1^i\} \le q_i \le \min\{1, Q_2^i\}, \\ \dfrac{\sigma(\rho_{i+1}^n) - d(F_i^{\text{in}}, l_i^n)}{1-\beta_i} & \text{if } 0 \le q_i < \max\{0, Q_1^i\}, \end{cases} \tag{4.11}$$

where

$$Q_1^i = \frac{\sigma(\rho_{i+1}^n) - d\left(F_i^{\text{in}}, l_i^n\right)}{\sigma(\rho_{i+1}^n)} \quad \text{and} \quad Q_2^i = \frac{(1-\beta_i)\delta(\rho_i^n)}{\sigma(\rho_{i+1}^n)}. \tag{4.12}$$

As in the previous case the dynamics of the buffer length is governed by

$$\frac{d}{dt}\, l_i^n = \begin{cases} F_i^{\text{in}} - (\sigma(\rho_i^n) - (1-\beta_i)\delta(\rho_i^n)) & \text{if } 1 \geq q_i > \min\{1, Q_2^i\}, \\ F_i^{\text{in}} - (1-q_i)\sigma(\rho_{i+1}^n) & \text{if } \max\{0, Q_1^i\} \leq q_i \leq \min\{1, Q_2^i\}, \\ F_i^{\text{in}} - d(F_i^{\text{in}}, l_i^n) & \text{if } 0 \leq q_i < \max\{0, Q_1^i\}, \end{cases}$$

which integrating yields

$$l_i^{n+1} = \begin{cases} l_i^n + \left(F_i^{\text{in}} - \left(\sigma(\rho_{i+1}^n) - (1-\beta_i)\delta(\rho_i^n)\right)\right)\Delta t & \text{if } 1 \geq q_i > \min\{1, Q_2^i\}, \\ l_i^n + \left(F_i^{\text{in}} - (1-q_i)\sigma(\rho_{i+1}^n)\right)\Delta t & \text{if } \max\{0, Q_1^i\} \leq q_i \leq \min\{1, Q_2^i\}, \\ l_i^n + \left(F_i^{\text{in}} - d(F_i^{\text{in}}, l_i^n)\right)\Delta t & \text{if } 0 \leq q_i < \max\{0, Q_1^i\}. \end{cases} \tag{4.13}$$

5 Instantaneous Optimization of ODE-PDE Constrained System

We are interested in determining discrete instantaneous optimal priority parameters $q_i^n = q_i(t^n)$ to minimize total travel time on the roundabout on a fixed time interval $[0, T]$, expressed by the cost functional

$$J(\vec{q}) = \sum_{i=1}^{m} \int_0^T \int_{I_i} \rho(t,x)\; dxdt + \sum_{i=1}^{m} \int_0^T l_i(t)\; dt. \tag{5.1}$$

Above, $\vec{q} : [0, \mathrm{T}] \to [0,1]^m$ is the time dependent vector of control variables. We select instantaneous optimal control approach to minimize the total travel time on the networks of the roundabout under consideration. Given a time step Δt (to be replaced with $\widetilde{\Delta} t$ when needed) compatible with the CFL condition (4.3), we introduce the instantaneous cost functional J^n at time t^n given as:

$$\begin{aligned} J^n(\vec{q}) &= \Delta t \sum_{i=1}^{m} L_i \rho_i^{n+1} + \Delta t \sum_{i=1}^{m} l_i^{n+1} \\ &= \Delta t \sum_{i=1}^{m} \left[L_i \rho_i^n - \Delta t \left(F_i^{n,-} - F_{i-1}^{n,+} \right) \right] \\ &\quad + \Delta t \sum_{i=1}^{m} \left[l_i^n + \Delta t \left(F_i^{\text{in}} - \gamma_{\mathrm{r}_i} \right) \right], \end{aligned} \tag{5.2}$$

where the dependencies of $F_i^{n,\pm}$ and γ_{r_i} on q_i are expressed by (4.7)-(4.9) in the demand-limited case and by (4.10)-(4.13) in the supply-limited situation. Within this framework, we treat both demand and supply-limited cases at junctions. In the computation of the gradient, we consider each cases based on the situation at the corresponding junction of the roundabout. The optimization problem thus writes

$$\min_{\vec{q} \in [0,1]^m} J^n(\vec{q}). \tag{5.3}$$

Note that in our case the cost functional J^n is piecewise linear with respect to q_i, $i = 1, \ldots, n$. Straight forward computations give

$$\frac{\partial}{\partial q_i^n} J^n(\vec{q}) = \begin{cases} \dfrac{\beta_i}{\beta_i - 1} \sigma(\rho_{i+1}^n)\Delta t^2 \leq 0 & \text{if } q_i \in [\max\{0, Q_1^i\}, \min\{1, Q_2^i\}] \\ 0 & \text{otherwise,} \end{cases} \tag{5.4}$$

where Q_1^i, Q_2^i are given by (4.12). J^n is decreasing on the interval $[Q_1^i, Q_2^i]$ and consequently the values of optimal priority parameter attained at Q_2^i. Therefore, any choice of $q_i \geq \min\{1, Q_2^i\}$, $i = 1, \ldots, m$ solves (5.3), resulting in the same flow dynamics. The role of the priority parameters is to force the priority to neither impose insufficient flows nor send excess vehicles than the carrying capacity of the main link of the roundabout. The optimal parameter values are used to update the priority variables q_i^n which adjusted for each iteration n to produce local optimal solution that satisfies the given target.

6 Numerical Tests

In this section we detail the results obtained through numerical simulations for a roundabout modeled as a concatenation of four 2×2 junctions with two incoming and two outing roads as depicted in Figure 5.

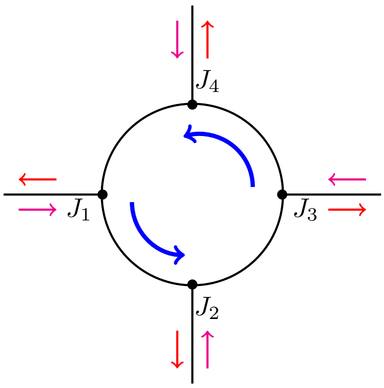

Figure 5: Roundabout considered for numerical simulations.

We represent the roundabout under consideration as:

- 5 roads forming the roundabout: I_1, I_2, I_3, I_4, I_5, with I_1 and I_5 linked with periodic boundary conditions;
- 4 roads connecting the roundabout with the rest of the road network: 4 incoming and 4 outgoing lanes.

We analyze the cost functional (5.1) introduced in Section 5. In particular, we want to compare the effectiveness of instantaneous optimal choices of the priority parameters given by $q_i = Q_2^i$, $i = 1, \ldots, m$, with respect to fixed constant parameters. For this, we compute the corresponding value of the discretized functional

$$\tilde{J}(\vec{q}) = \Delta t \sum_{n=0}^{n_T} \sum_{i=1}^{m} L_i \rho_i^{n+1} + \Delta t \sum_{n=0}^{n_T} \sum_{i=1}^{m} l_i^{n+1}, \tag{6.1}$$

with $\vec{q} : [0, T] \to [0, 1]^m$ is piecewise constant on $[t^n, t^{n+1}[$, $n = 0, \ldots, n_T$,where we set $t^0 = 0$ and $t^{n_T+1} = T$.

6.1 Simulation Characteristics

Below, we present the choice of simulation parameters. The numerical approximation of the hyperbolic conservation laws that describe the evolution of densities for each road of the roundabout is made using the adapted Godunov scheme described in Section 4.3. The time step is determined by the CFL stability condition (4.3) with coefficient equal to 0.5. Periodic boundary conditions are imposed on the left and on the right of the computational domain. The ODE is discretized using an explicit Euler first order integration method. The roundabout traffic evolution is simulated on a time interval $[0, T]$, with $T = 30$, long enough to attain a stabilized situation.

At initial time $t = 0$, we assume that all the links and the buffers are empty, and we impose $\mathrm{F_i}^{\mathrm{in}} \neq 0$ at each junctions. For the simulations, we consider $\rho_{max} = v_f = 1$, $f^{\max} = \rho_c = 0.66$, $L_i = 1$ and $\gamma_{\mathrm{r}_i}^{\max} = 0.65$ for all $i = 1, \ldots, 4$. Finally, for each value of the fixed control parameters we study 6×6 simulation cases based on different values of the flux $\mathrm{F_i}^{\mathrm{in}}$ and β_i. For each of them we perform two simulations:

1. **Instantaneous optimal priority parameters:** We use $q_i = Q_2^i$, $i = 1, \ldots, 4$ in Algorithm 1 to compute the corresponding value of (6.1).

2. **Fixed optimal priority parameters:** In this case, the priority parameters q_i are kept fixed with the same constant values for all junctions in each simulation cases. Then, as in case 1, we apply Algorithm 1 to compute the state variables on each links of the roundabout and the buffer lengths and the corresponding total travel time on the roundabout.

6.2 Simulation Results

We run simulations for the values of $\mathrm{F_i}^{\mathrm{in}}$ and β_i as given in Tables 1-6. Then we compute the difference between total travel time obtained with different fixed q_i and the optimal q_i^n. That is

$$\mathrm{TTTD} = \frac{\mathrm{TTTF} - \mathrm{TTTO}}{\mathrm{TTTF}}$$

where TTTF stands for the value of (6.1) corresponding to fixed q_i, TTTO represents total travel time with optimal q_i^n and TTTD for their difference. Tables 1-6 report the gain percentage in total travel times to compare the effectiveness of our approach.

$\beta_i \backslash \mathrm{F_i}^{\mathrm{in}}$	0.1	0.2	0.3	0.4	0.5	0.6
0.2	0%	47.42%	36.86%	29.44%	24.42%	20.66%
0.3	0%	0%	54.12%	43.42%	36.08%	30.52%
0.4	0%	0%	69.87%	57.42%	47.68%	40.36%
0.5	0%	0%	0%	71.07%	59.3%	50.11%
0.6	0%	0%	0%	0%	70.75%	59.76%
0.7	0%	0%	0%	0%	81.68%	68.92%

Table 1: Improvement in total travel time using instantaneous optimal parameters compared to fixed constant parameter $q_i = 0.2, i = 1, \ldots, 4$.

β_i \ F_i^{in}	0.1	0.2	0.3	0.4	0.5	0.6
0.2	0%	47.3%	36.76%	29.38%	24.15%	20.35%
0.3	0%	0%	54.04%	43.24%	35.72%	29.99%
0.4	0%	0%	69.72%	57.23%	47.19%	39.57%
0.5	0%	0%	0%	70.63%	58.4%	48.92%
0.6	0%	0%	0%	0%	69.01%	57.2%
0.7	0%	0%	0%	0%	0%	0%

Table 2: Percentage gained in total travel time using instantaneous optimal parameters compared to fixed constant parameter $q_i = 0.3, i = 1, \ldots, 4$.

β_i \ F_i^{in}	0.1	0.2	0.3	0.4	0.5	0.6
0.2	0%	47.19%	36.6%	29.14%	23.74%	19.94%
0.3	0%	0%	53.7%	42.95%	34.97%	29.32%
0.4	0%	0%	69.22%	56.51%	45.75%	38.29%
0.5	0%	0%	0%	69.11%	55.36%	45.86%
0.6	0%	0%	0%	0%	0%	0%
0.7	0%	0%	0%	0%	0%	0%

Table 3: Percentage gain in total travel time using instantaneous optimal parameters compared to fixed constant parameter $q_i = 0.4, i = 1, 2, \ldots, 4$.

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β_i \ F_i^{in}	0.1	0.2	0.3	0.4	0.5	0.6
0.2	0%	46.94%	36.25%	28.36%	23.07%	19.38%
0.3	0%	0%	53.04%	41.49%	33.55%	28.02%
0.4	0%	0%	67.41%	52.97%	42.27%	35.03%
0.5	0%	0%	0%	0%	0%	0%
0.6	0%	0%	0%	0%	0%	0%
0.7	0%	0%	0%	0%	0%	0%

Table 4: Percentage gain in total travel time using instantaneous optimal parameters compared to fixed constant parameter $q_i = 0.5, i = 1, 2, \ldots, 4$.

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β_i \ F_i^{in}	0.1	0.2	0.3	0.4	0.5	0.6
0.2	0%	46.22%	34.95%	27.02%	21.91%	18.33%
0.3	0%	0%	49.42%	37.67%	30.12%	24.96%
0.4	0%	0%	0%	0%	0%	0%
0.5	0%	0%	0%	0%	0%	0%
0.6	0%	0%	0%	0%	0%	0%
0.7	0%	0%	0%	0%	0%	0%

Table 5: Percentage gain in total travel time using instantaneous optimal parameters compared to fixed parameter $q_i = 0.6, i = 1, 2, \ldots, 4$.

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β_i \ F_i^{in}	0.1	0.2	0.3	0.4	0.5	0.6
0.2	0%	43.23%	30.99%	23.7%	19.07%	15.87%
0.3	0%	0%	0%	0%	0%	10%
0.4	0%	0%	0%	0%	0%	0%
0.5	0%	0%	0%	0%	0%	0%
0.6	0%	0%	0%	0%	0%	0%
0.7	0%	0%	0%	0%	0%	0%

Table 6: Percentage gain in total travel time using instantaneous optimal parameters compared to fixed constant parameters $q_i = 0.7, i = 1, 2, \ldots, 4$.

As indicated in Tables 1-6, the percentage gain in total travel time is nul in the case of demand-limited and both approaches give the same value. However, as the traffic inflow increases on the incoming secondary road at the entrance of the roundabout, a sensible percentage gain is observable for different fixed constant q_i and optimal q_i, as illustrated in the above tables. These situation corresponds to supply-limited cases on the main lane of the roundabout. This shows that, when a large population of vehicles remains on the main road of the roundabout, the instantaneous optimal choice of priority parameter increases the performance of the roundabout compared to different fixed values of q_i.

For the fixed priority parameters greater than Q_2^i at each junction both approaches take the same values. This is due to the fact illustrated by Figure 4c in Section 3. That is, the junction solution is approximated by the point on the feasible set. Furthermore, as the values of the fixed parameters q_i decreases, the instantaneous choice of optimal parameter shows better performance in increasing traffic throughput. Compared to small fixed priority parameters, larger fixed values are better in optimizing traffic flux on the network portion.

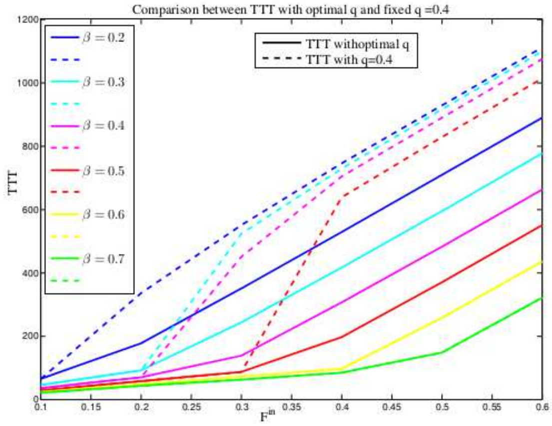

Figure 6: Comparison between the total travel times obtained with optimal choice of q_i and fixed $q_i = 0.4$, for different values of incoming flux and splitting ratio.

In Figure 6 we plot the values of total travel time obtained by both approaches as a function of the incoming flux $\mathrm{F_i}^{\mathrm{in}}$ for different values of splitting ratio β_i for the whole roundabout. The legend indicates different simulation cases.

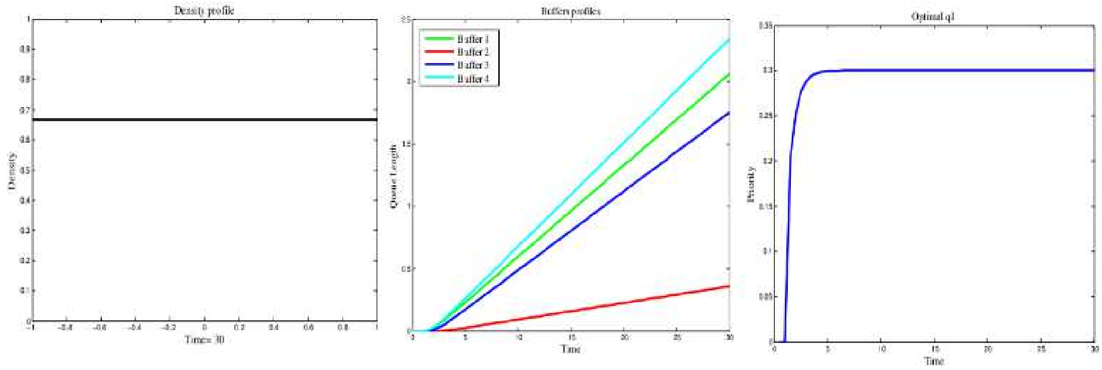

Figure 7: Density at $T = 30$ (left), buffer lengths (middle) and optimal priority parameter (right) evolution on the time interval $[0, 30]$.

Figure 7 illustrates sample density, buffer length and optimal priority profiles for different inflows from the secondary incoming road of a roundabout at junctions during the simulation time interval $[0, 30]$.

7 Conclusion

In this paper, we present an instantaneous traffic flow optimization approach on a roundabout using scalar conservation laws. We first discretize the hyperbolic PDE via the Godunov scheme and the ODE using an explicit Euler first order integration method. We compute the gradient of the cost functional that measures the instantaneous total travel time at each time step t^n and find the optimal priority parameters. Numerical tests are conducted for a roundabout with four incoming and four outgoing roads. The simulation results show that gradient based instantaneous optimal choice of the priority parameters produces beneficial results in improving the performance of the roundabout compared to the fixed choice of priority parameters. In the case of fixed priority parameters, larger fixed values are better in optimizing traffic flux on the main lane of the roundabout relative to smaller values.

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